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The Fate of Lifshits Tails in Magnetic Fields

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We investigate the integrated density of states of the Schrödinger operator in the Euclidean plane with a perpendicular constant magnetic field and a random potential. For a Poisson random potential with a nonnegative, algebraically decaying, single-impurity potential we prove that the leading asymptotic behavior for small energies is always given by the corresponding classical result, in contrast to the case of vanishing magnetic field. We also show that the integrated density of states of the operator restricted to the eigenspace of any Landau level exhibits the same behavior. For the lowest Landau level, this is in sharp contrast to the case of a Poisson random potential with a delta-function impurity potential.

KEY WORDS: Random Schrödinger operators; magnetic fields; Lifshits tails.

1. INTRODUCTION

Random Schrödinger operators are operators on $L^2(\mathbb{R}^d)$ formally given by $-\frac{1}{2}\nabla^2 + V_\omega$, where V_ω is an ergodic (or metrically transitive) random scalar potential. Here ∇ denotes the nabla operator in the d-dimensional Euclidean space $\mathbb{R}^d$ and $L^2(\mathbb{R}^d)$ is the Hilbert space of Lebesgue square-integrable complex-valued functions on $\mathbb{R}^d$. These operators have been thoroughly investigated by physicists as well as mathematicians. See refs. 38, 11, and 23 for reviews of basic concepts and rigorous results. For a more physical point of view see refs. 43 and 31. The generalizations of

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random Schrödinger operators to nonzero constant magnetic field are operators of the form

$$H(V_\omega) := \tfrac{1}{2}(i\nabla + A)^2 + V_\omega \tag{1.1}$$

where A: $\mathbb{R}^d \to \mathbb{R}^d$ is a nonrandom vector potential such that the magnetic field tensor (B_{jk}) given by $B_{jk} := \partial A_j/\partial x_k - \partial A_k/\partial x_j$ is constant. The two-dimensional version of (1.1) is widely believed to serve as a minimal model for the integer quantum Hall effect[48, 21, 2] and has therefore been intensively investigated by physicists; see, for instance, refs. 1, 26, 48, and 21.

Only recently rigorous studies of the spectral properties of random operators of the form (1.1) have appeared[34, 35, 8, 49, 50, 12, 16, 17, 10] (but see also the related earlier work of ref. 19). The self-averaging property of the integrated density of states has been established under fairly general conditions and various of its asymptotic properties have been considered. The existence of localized states also has been proven.[12, 16, 17]

In this paper we are concerned with the Schrödinger operator (1.1) in two dimensions with a Poisson random potential determined by a non-negative single-site potential U, decaying algebraically at infinity. That is,

$$V_\omega(x) = \sum_j U(x - p_\omega(j)) \tag{1.2}$$

where $p_\omega(j)$ are random points in the Euclidean place $\mathbb{R}^2$ distributed in accordance with Poisson's law. We will show how rigorous versions of results in refs. 6 and 7 can be used to find the leading asymptotic behavior of the integrated density of states as the energy approaches the infimum of the spectrum from above.

In the zero-field case the asymptotic behavior changes its character depending on the decay of the single-site potential at infinity. For slow decay it is governed solely by the potential energy,[37] that is, by classical effects. For rapid decay the kinetic energy also becomes relevant, leading to genuine quantum effects. The form of the latter asymptotic behavior was discovered by Lifshits.[29–31] Convincing arguments for the validity of Lifshits' conjecture were given by Friedberg and Luttinger.[18, 32] Its rigorous proof[15, 36, 37] relies on Donsker and Varadhan's[15] involved large-deviation results (see also ref. 14, Section 4.3) about the long-time asymptotics of certain Wiener integrals.

For the case of nonzero constant magnetic field we will show that the low-energy tail is always given by the classical result irrespective of the decay of the single-site potential. Basically, this is due to the fact that in this case the ground state of the unperturbed Schrödinger operator $H(0)$ consists of square-integrable functions.

The distance between successive eigenvalues of $H(0)$, commonly referred to as Landau levels, and the degeneracy per area of each eigenvalue increase linearly with the strength of the magnetic field. For a fixed concentration of noninteracting electrons and sufficiently strong field it is therefore physically reasonable to investigate only the restriction $E_0 H(V_\omega) E_0$ to the eigenspace $E_0 \mathrm{L}^2(\mathbb{R}^2)$ of the lowest Landau level instead of the full Schrödinger operator $H(V_\omega)$. For a rigorous discussion of this point see refs. 34, 8, and 10. Remarkably, Wegner[51] succeeded in calculating the corresponding restricted integrated density of states for the case of a delta-correlated Gaussian random potential. A rigorous version of this derivation is given in ref. 34. Wegner's result was quickly extended to general delta-correlated random potentials by Brézin *et al.*,[5] including Poisson potentials, the single-site potential of which is a Dirac delta function. The calculation in ref. 5 relies on the resummation of a suitable representation of an averaged Neumann series. A nonperturbative derivation was given by Klein and Perez.[25]

We will show that the restriction to any Landau level does not alter the leading asymptotic behavior of the integrated density of states at the lower spectral boundary for algebraically decaying single-site potentials. This behavior is in sharp contrast to that for delta-correlated Poisson potentials, where it can happen that the integrated density of states is not continuous at the infimum of the spectrum.

Although we will only discuss the two-dimensional case, our result generalizes to even dimensions and a nondegenerate magnetic field tensor, that is, to the case where (B_{jk}) is constant and has full rank.

2. STATEMENT OF THE RESULT

We will assume throughout that the random potential V_ω is the *Poisson random potential* [ref. 38, Example 1.15(d)] with concentration $\varrho > 0$ and *nonnegative single-site potential* U: $\mathbb{R}^2 \to [0, \infty]$, where U decays algebraically and integrably at infinity, that is,

$$U \geqslant 0, \qquad \lim_{|x| \to \infty} |x|^\alpha \, U(x) = \mu, \qquad 0 < \mu < \infty, \quad \alpha > 2 \tag{2.1}$$

In addition, for technical reasons, we cannot allow too strong local singularities. Therefore we will assume besides (2.1) either local square integrability

$$U \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2) \tag{2.2}$$

or, more restrictively, local boundedness

$$U \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^2) \tag{2.3}$$

Remark 2.1. (i) Either of the conditions ensures that V_ω is an ergodic, measurable, nonnegative random potential, where ergodicity is meant with respect to the group of spatial translations in $\mathbb{R}^2$.

(ii) We note that (2.1) together with (2.2) implies

$$\|U\|_p < \infty \qquad \text{for all} \quad 1 \leqslant p \leqslant 2 \tag{2.4}$$

while (2.1) together with (2.3) implies

$$\|U\|_p \leqslant (\|U\|_1)^{1/p} (\|U\|_\infty)^{(p-1)/p} < \infty \qquad \text{for all} \quad 1 \leqslant p \leqslant \infty \tag{2.5}$$

Here, as usual

$$\|f\|_p := \begin{cases} \left[\int |f(y)|^p \, dy\right]^{1/p} & \text{for} \quad p < \infty \\ \operatorname*{ess\,sup}_{y \in \mathbb{R}^2} |f(y)| & \text{for} \quad p = \infty \end{cases} \tag{2.6}$$

denotes the norm of $f \in \mathrm{L}^p(\mathbb{R}^2)$.

Let $\mathbb{E}$ denote the expectation with respect to the random potential. By (2.4) or (2.5) one has the finiteness

$$\mathbb{E}[(V_\omega(x))^2] = \varrho \int [U(y)]^2 \, dy + \left[\varrho \int U(y) \, dy\right]^2 < \infty \tag{2.7}$$

of the potential's second moment at any point $x \in \mathbb{R}^2$. This implies that the potential V_ω is almost surely locally square-integrable and, moreover, that the mapping

$$x \mapsto \int_{|x-y| \leqslant 1} [V_\omega(y)]^2 \, dy \tag{2.8}$$

is almost surely polynomially bounded as $|x| \to \infty$ (confer ref. 23, Proof of Theorem 5.1).

By (2.5) one even has the finiteness

$$\begin{aligned} \mathbb{E}[(V_\omega(x))^k] &\leqslant k!\,(\|U\|_\infty)^k \left\{\mathbb{E}\left[\exp\left(\frac{V_\omega(x)}{\|U\|_\infty}\right)\right] - 1\right\} \\ &= k!\,(\|U\|_\infty)^k \left(\exp\left\{\varrho \int \left[\exp\left(\frac{U(y)}{\|U\|_\infty}\right) - 1\right] dy\right\} - 1\right) \\ &\leqslant k!\,(\|U\|_\infty)^k \left\{\exp\left[\varrho \frac{\|U\|_1}{\|U\|_\infty}(e-1)\right] - 1\right\} < \infty \end{aligned} \tag{2.9}$$

of the kth moment for all positive integers $k \in \mathbb{N}$.

(iii) The Laplace characteristic functional of V_ω

$$\mathbb{E}\left[\exp\left(-\int J(x)\, V_\omega(x)\, dx\right)\right]$$
$$= \exp\left\{-\varrho \int \left[1 - \exp\left(-\int J(x-y)\, U(y)\, dy\right)\right] dx\right\} \tag{2.10}$$

is well defined for all nonnegative $J \in \mathscr{S}(\mathbb{R}^2)$ (confer, e.g., ref. 38, Proposition 1.16). Here we have introduced $\mathscr{S}(\mathbb{R}^2)$ for the Schwartz space of rapidly decreasing, arbitrarily-often-differentiable functions on $\mathbb{R}^2$.

(iv) One can look upon V_ω as given by

$$V_\omega(x) = \int U(x-y)\, dm_\omega(y) \tag{2.11}$$

where m_ω is the Poisson random measure on $\mathbb{R}^2$ with concentration ϱ. Since m_ω is purely atomic, V_ω can be interpreted physically as the potential generated by uniformly distributed noninteracting impurities, the influence of each impurity being described by the same repulsive potential U; see (1.2).

In the sequel we suppose $B > 0$ and the vector potential

$$A(x) := \frac{B}{2}\begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}, \qquad x =: \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{2.12}$$

given in the symmetric gauge. Then the *Schrödinger operator* (1.1) is almost surely essentially self-adjoint on $\mathscr{S}(\mathbb{R}^2)$ provided (2.1) and (2.2) hold. This follows from ref. 13, Theorem 1.15, and the fact that $V_\omega \psi \in \mathrm{L}^2(\mathbb{R}^2)$ for all $\psi \in \mathscr{S}(\mathbb{R}^2)$ almost surely because the mapping (2.8) is polynomially bounded. On account of gauge equivalence,[28] with the choice (2.12) there is no loss of generality for the description of a constant magnetic field.

The spectral resolution of the unperturbed part dates back to Landau[27] and is given by

$$H(0) = \sum_{n=0}^{\infty} \varepsilon_n E_n, \qquad \varepsilon_n := (n + \tfrac{1}{2})\, B \tag{2.13}$$

where the eigenvalues ε_n are called Landau levels and the corresponding infinite-dimensional eigenprojectors E_n are integral operators with kernels given by

$$E_n(x, y) := \frac{B}{2\pi} \exp\left[\frac{iB}{2}(x_1 y_2 - x_2 y_1) - \frac{B}{4}(x-y)^2\right] \mathrm{L}_n\left(\frac{B}{2}(x-y)^2\right) \tag{2.14}$$

Here L_n is the nth Laguerre polynomial (ref. 20, Section 8.97).

The explicit formula for the integral kernel of E_n shows that $E_n(x, \cdot) \in \mathscr{S}(\mathbb{R}^2)$. Since $\mathscr{S}(\mathbb{R}^2)$ is stable under convolution, this implies that $\mathscr{S}(\mathbb{R}^2) \cap E_n \mathrm{L}^2(\mathbb{R}^2) \subset E_n \mathscr{S}(\mathbb{R}^2) \subset \mathscr{S}(\mathbb{R}^2)$. Furthermore, $E_n \mathscr{S}(\mathbb{R}^2)$ is dense in $E_n \mathrm{L}^2(\mathbb{R}^2)$, because $\mathscr{S}(\mathbb{R}^2)$ is dense in $\mathrm{L}^2(\mathbb{R}^2)$. Taking finally into account that the mapping (2.8) is almost surely polynomially bounded, we conclude that the Schrödinger operator restricted to the eigenspace of the nth Landau level, in symbols $E_n H(V_\omega) E_n = \varepsilon_n E_n + E_n V_\omega E_n$, and all its natural powers $(E_n H(V_\omega) E_n)^k$, $k \in \mathbb{N}$, are almost surely well-defined nonnegative operators from $\mathscr{S}(\mathbb{R}^2)$ to $\mathrm{L}^2(\mathbb{R}^2)$.

As for the essential self-adjointness of $E_n H(V_\omega) E_n$ on $\mathscr{S}(\mathbb{R}^2)$, we were not able to prove it under the assumptions (2.1) and (2.2). However, under the stronger assumptions (2.1) and (2.3) a proof can be tailored along the lines given in the proof of Theorem 2.1 in ref. 16. By appealing to Nelson's analytic-vector theorem (ref. 41, Theorem X.39) and to the fact that $\mathscr{S}(\mathbb{R}^2)$ contains a countable dense subset it suffices to check that

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} \{\mathbb{E}[(\|(E_n V_\omega E_n)^k \psi\|_2)^2]\}^{1/2} < \infty \tag{2.15}$$

for all $\psi \in \mathscr{S}(\mathbb{R}^2)$ and some $t > 0$. We start by observing that there is a constant $c_n(B)$ such that

$$|E_n(x, y)| \leqslant c_n(B) \frac{B}{8\pi} e^{-(B/8)(x-y)^2} \tag{2.16}$$

for all $x, y \in \mathbb{R}^2$. Moreover, by an iterated version of Hölder's inequality and translation invariance one has

$$\mathbb{E}\left[\prod_{j=1}^{2k} V_\omega(x^{(j)})\right] \leqslant \mathbb{E}[(V_\omega(0))^{2k}] \tag{2.17}$$

for all $x^{(1)}, \ldots, x^{(2k)} \in \mathbb{R}^2$. Copying now the steps between Eqs. (2.15) and (2.20) in ref. 16, we find that our estimates (2.16), (2.17), and (2.9) yield for all $k \in \mathbb{N}$

$$\mathbb{E}[(\|(E_n V_\omega E_n)^k \psi\|_2)^2] \leqslant (2k)!\, [c_n(B) \|U\|_\infty]^{2k} \times \frac{B c_n(B)(\|\psi\|_1)^2}{8\pi(2k+1)} \left\{\exp\left[\varrho \frac{\|U\|_1}{\|U\|_\infty}(e-1)\right] - 1\right\} \tag{2.18}$$

As a consequence, (2.15) is valid for all $0 \leqslant t < [2c_n(B) \|U\|_\infty]^{-1}$.

In the following we are allowed and will understand the unrestricted operator $H(V_\omega)$ and the restricted operators $E_n H(V_\omega) E_n$ as to be almost surely essentially self-adjoint on $\mathscr{S}(\mathbb{R}^2)$ under the imposed assumptions (2.1), (2.2) and (2.1), (2.3), respectively.

Let $\mathbb{R} \ni \lambda \mapsto P_\lambda(X) = \Theta(\lambda - X)$ denote the projection-valued measure for the self-adjoint operator X. Here Θ is Heaviside's unit-step function: $\Theta(a) = 0$ for $a < 0$ and $\Theta(a) = 1$ for $a \geqslant 0$. Our objects of study, the *integrated density of states* N and the *nth restricted integrated density of states* R_n, are defined by

$$N(\lambda) := \mathbb{E}[P_\lambda(H(V_\omega))(x, x)] \tag{2.19}$$

and

$$R_n(\lambda) := \mathbb{E}[(E_n P_\lambda(E_n H(V_\omega) E_n) E_n)(x, x)] \tag{2.20}$$

respectively.

Remark 2.2. (i) The integral kernel $(x, y) \mapsto P_\lambda(H(V_\omega))(x, y)$ of the spectral projection $P_\lambda(H(V_\omega))$ almost surely exists and is jointly continuous in (x, y). This can be seen from ref. 9, Section 6, using Remark 2.1(ii). For vanishing vector potential this result is standard (ref. 47, Theorem B.7.1). The above continuity ensures that (2.19) is well defined. Since V_ω is translation invariant, the right-hand side of (2.19) is independent of $x \in \mathbb{R}^2$.

(ii) In ref. 49, Proposition 3.2, and ref. 10 it is shown that the more familiar and physically reasonable way of defining the integrated density of states by means of a macroscopic limit yields (2.19). This amounts to first restricting the Schrödinger operator $H(V_\omega)$ to a finite region—a square with Dirichlet boundary conditions, say. Then one defines the integrated density of states of the finite system to be the number of eigenvalues below λ divided by the region's area. Finally, one proves that this quantity becomes nonrandom in the macroscopic limit, which is usually summarized as the self-averaging property of the integrated density of states.

(iii) The Cauchy–Schwarz inequality, $E_n^2 = E_n$, and $E_n(x, x) = B/(2\pi)$ yield for all $\psi \in \mathrm{L}^2(\mathbb{R}^2)$

$$\|E_n \psi\|_\infty \leqslant \left(\frac{B}{2\pi} \langle \psi, \psi \rangle\right)^{1/2} = \left(\frac{B}{2\pi}\right)^{1/2} \|\psi\|_2 \tag{2.21}$$

where, as usual, $\langle \cdot, \cdot \rangle$ symbolizes the standard scalar product for $\mathrm{L}^2(\mathbb{R}^2)$. Therefore, the restriction $E_n X E_n$ of any bounded self-adjoint operator X on $\mathrm{L}^2(\mathbb{R}^2)$ to the eigenspace of the nth Landau level is a Carleman integral operator (ref. 47, Corollary A.1.2). Since $E_n(x, y)$ is smoothing and a projection, the integral kernel $(x, y) \mapsto (E_n X E_n)(x, y)$ is jointly continuous in (x, y). Hence (2.20) is well defined, too. Furthermore, the right-hand side of (2.20) is independent of $x \in \mathbb{R}^2$ due to the translation invariance of V_ω; see the Appendix for details.

(iv) On physical grounds R_0 should be a reasonable approximation to N for strong magnetic fields. This is given a precise meaning in ref. 34, Proposition 1, ref. 8, Theorem 5, and ref. 10. For the significance of R_n for general n see refs. 7 and 8.

(v) In the reasoning in the above remarks (i) and (iii) we have swept measurability questions with respect to ω under the rug, as we will do in the sequel. These problems, however, can be fixed by the methods of ref. 24 or ref. 11, Sections V.1 and V.3.

Our aim is to identify the asymptotics of $N(\lambda)$ and $R_n(\lambda)$ as λ approaches the lower spectral boundary, that is, $\lambda \downarrow \inf \operatorname{spec}(H(V_\omega))$ or $\lambda \downarrow \inf \operatorname{spec}(E_n H(V_\omega) E_n)$, respectively. By employing the so-called magnetic translations[52, 25] [see Eq. (A.17) in the Appendix], one can show by standard arguments[24, 23] that both $H(V_\omega)$ and $E_n H(V_\omega) E_n$ are ergodic families of operators.[49, 50, 12, 16, 10] Therefore, their spectra are nonrandom quantities; see ref. 49, Theorem 2.1, ref. 24, Theorem 1, or ref. 23, Theorem 4.3.1. It will turn out that $\inf \operatorname{spec}(H(V_\omega)) = \varepsilon_0$ and $\inf \operatorname{spec}(E_n H(V_\omega) E_n) = \varepsilon_n$, as expected. Our result is the following.

Theorem 2.3. Under the assumptions (2.1) and (2.2) one has for all $B > 0$

$$\lim_{\lambda \downarrow 0} \lambda^{2/(\alpha-2)} \ln N(\varepsilon_0 + \lambda) = -C(\alpha, \mu, \varrho) \tag{2.22}$$

for the integrated density of states N and, similarly, under the assumptions (2.1) and (2.3)

$$\lim_{\lambda \downarrow 0} \lambda^{2/(\alpha-2)} \ln R_n(\varepsilon_n + \lambda) = -C(\alpha, \mu, \varrho) \tag{2.23}$$

for the nth restricted integrated density of states R_n, $n \in \mathbb{N} \cup \{0\}$. Here we have set

$$C(\alpha, \mu, \varrho) := \frac{1}{2} (\alpha - 2)\, \mu^{2/(\alpha-2)} \left[\frac{2\pi\varrho}{\alpha}\, \Gamma\left(\frac{\alpha-2}{\alpha}\right) \right]^{\alpha/(\alpha-2)} > 0 \tag{2.24}$$

where Γ denotes Euler's gamma function.

Remark 2.4. (i) The result for the unrestricted integrated density of states N should be compared with the case $B = 0$. The asymptotic decay at the lower spectral boundary coincides with the behavior for $B=0$ if $(d=)\ 2<\alpha<4\ (=d+2)$, that is, for a slowly decaying single-site potential (see ref. 37 or ref. 38, Corollary 9.14), but it differs in the case $\alpha>4$ $(=d+2)$ (see refs. 36 and 37, or ref. 38, Theorem 10.2). This is plausible from the Rayleigh–Ritz-like variational principle due to Luttinger[32] and Pastur[37]; see also Eq. (17.3) and Chapter 21 in ref. 31. For $B=0$ and $\alpha>4$ the optimal wavefunction becomes too sharply localized so that the unperturbed (kinetic) energy begins to play a significant role. For $B>0$ the contribution of the unperturbed energy relative to the ground-state energy ε_0 can always be kept zero by choosing one of the square-integrable ground-state wavefunctions of $H(0)$.

(ii) The lowest restricted integrated density of states R_0 for a delta-correlated Poisson potential, corresponding to $U(x)=v\delta(x)$, $v>0$, is known exactly,[5,25] not only in the low-energy tail. If the mean number $2\pi\varrho/B$ of impurities over the spatial extent πl^2 of the ground-state wavefunctions of $H(0)$,

$$l^2 := \frac{1}{E_0(y,y)} \int E_0(y,x)(x-y)^2\, E_0(x,y)\, dx = \frac{2}{B} \tag{2.25}$$

is smaller than one, R_0 exhibits a jump at ε_0, the height of which is proportional to $1-2\pi\varrho/B$. This is plausible, because in the case $2\pi\varrho/B<1$ one can imagine that, effectively, only a fraction of the ground-state wavefunctions are affected by the impurities.[5] This reasoning does not seem applicable to algebraically decaying impurity potentials, in accordance with our result that R_n is continuous at ε_n.

(iii) For the reader's convenience, we present the generalizations of (2.22) and (2.23) to even space dimensions $d\geqslant 2$:

$$\begin{aligned}
&\lim_{\lambda\downarrow 0} \lambda^{d/(\alpha-d)} \ln N(\varepsilon_0+\lambda)\\
&\quad = \lim_{\lambda\downarrow 0} \lambda^{d/(\alpha-d)} \ln R_n(\varepsilon_n+\lambda)\\
&\quad = -\frac{\alpha-d}{d}\, \mu^{d/(\alpha-d)} \left[\frac{\varrho}{\alpha}\,\frac{2\pi^{d/2}}{\Gamma(d/2)}\,\Gamma\!\left(\frac{\alpha-d}{\alpha}\right)\right]^{\alpha/(\alpha-d)}
\end{aligned} \tag{2.26}$$

Here it is assumed that $\alpha>d$ and that the magnetic field tensor (B_{jk}) is constant and has full rank.

(iv) It would be interesting to compute subleading corrections to (2.26) as Luttinger and Waxler[33] did for zero magnetic field and, of course, $d<\alpha<d+2$. In contrast to the leading term given by (2.26), we expect these corrections to depend on the field.

3. PROOF

For the proof of Theorem 2.3 we follow the strategy in ref. 37. Instead of $N(\lambda)$ we investigate its Laplace transform and use a Tauberian argument (ref. 38, Theorem 9.7). Since $H(V_\omega)$ is bounded below by ε_0, this argument shows that (2.22) is equivalent to

$$\lim_{t\to\infty} t^{-2/\alpha} \ln \tilde{N}(t) = -\pi\varrho\mu^{2/\alpha}\Gamma\left(\frac{\alpha-2}{\alpha}\right) = -\varrho\mu^{2/\alpha}\int (1-e^{-|x|^{-\alpha}})\,dx \qquad (3.1)$$

where

$$\tilde{N}(t) := \int e^{-t\lambda}\,dN(\varepsilon_0+\lambda) = e^{t\varepsilon_0}\int e^{-t\lambda}\,dN(\lambda), \qquad t>0 \qquad (3.2)$$

is the shifted Laplace transform of N. Analogously we define

$$\tilde{R}_n(t) := \int e^{-t\lambda}\,dR_n(\varepsilon_n+\lambda) = e^{t\varepsilon_n}\int e^{-t\lambda}\,dR_n(\lambda), \qquad t>0 \qquad (3.3)$$

to be the shifted Laplace transform of R_n. Then (2.23) is equivalent to (3.1) with $\tilde{R}_n$ replacing $\tilde{N}$.

To establish the leading asymptotic behavior of $\tilde{N}(t)$ and $\tilde{R}_n(t)$ for large t we use the following preparatory results.

Basic Inequalities 3.1. Let

$$\phi_n(\cdot) := \left(\frac{2\pi}{B}\right)^{1/2} E_n(\cdot, 0) \qquad (3.4)$$

Then

$$\frac{1}{2\pi t}\,\mathbb{E}[e^{-t\langle\phi_0,\, V_\omega\phi_0\rangle}] \leqslant \tilde{N}(t) \qquad (3.5)$$

$$\tilde{N}(t) \leqslant \frac{Be^{t\varepsilon_0}}{4\pi\sinh(t\varepsilon_0)}\,\mathbb{E}[e^{-tV_\omega(0)}] \qquad (3.6)$$

and

$$\frac{B}{2\pi}\,\mathbb{E}[e^{-t\langle\phi_n,\, V_\omega\phi_n\rangle}] \leqslant \tilde{R}_n(t) \qquad (3.7)$$

$$\tilde{R}_n(t) \leqslant \frac{B}{2\pi}\,\mathbb{E}[e^{-tV_\omega(0)}] \qquad (3.8)$$

Remark 3.2. (i) One can infer from (2.14) that $\|\phi_n\|_2 = 1$ and $\phi_n \in \mathscr{S}(\mathbb{R}^2)$. Therefore, the left-hand sides of (3.5) and (3.7) are well defined, because almost surely $V_\omega \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)$ and the mapping (2.8) is polynomially bounded.

(ii) The bound (3.6) is a generalization of the classical upper bound for $\tilde{N}$ (ref. 37; ref. 38, Theorem 9.1) to nonzero magnetic fields. The essential ingredient for its derivation is the Golden–Thompson inequality. The bounds (3.7) and (3.8) stem from the Jensen–Peierls inequality. The inequality (3.5) is a variant of an inequality due to Berezin, Lieb, and Luttinger, which in turn follows from the Jensen–Peierls inequality. In ref. 37 and ref. 38, Theorem 9.5, a lower bound similar to (3.5) is proven. It has the advantage of holding for more general families of random operators, but allows for functions ϕ_0 with compact support only. For our proof it is essential, however, to allow for ϕ_0 as given in (3.4) with unbounded support.

(iii) Nonrigorous derivations of the above bounds can be found in refs. 6 and 7.

(iv) A rigorous derivation of (3.6) for the case of a Gaussian random potential with a Gaussian covariance function can be read off from Eqs. (2.3), (2.8), and (2.10) in ref. 34.

(v) The above bounds are to some extent special cases of those presented in ref. 10 for potentials more general than Poisson random potentials. Our lines of reasoning are closely related to those in ref. 10, but are considerably simplified by the fact that $V_\omega \geqslant 0$ almost surely. We have banished the proofs of the Basic Inequalities to the Appendix because on the one hand they follow the plan of refs. 6 and 7 and on the other hand there are some unwieldy technicalities involved in supplying the missing rigor.

The Basic Inequalities provide us with asymptotically coinciding upper and lower bounds for the shifted Laplace transforms $\tilde{N}$ and $\tilde{R}_n$.

Upper Bound. The inequality (3.8) and Remark 2.1(iii) imply

$$\tilde{R}_n(t) \leqslant \frac{B}{2\pi} \exp\left[-\varrho \int (1 - e^{-tU(x)})\, dx \right] \tag{3.9}$$

Therefore we have

$$\limsup_{t\to\infty} t^{-2/\alpha} \ln \tilde{R}_n(t) \leqslant -\varrho \liminf_{t\to\infty} t^{-2/\alpha} \int (1 - e^{-tU(x)})\, dx \tag{3.10}$$

On the right-hand side of (3.10) we substitute $x \mapsto (\mu t)^{1/\alpha} x$ and use Fatou's lemma to interchange the limit and the integration. Since

$$\lim_{t \to \infty} tU((\mu t)^{1/\alpha} x) = |x|^{-\alpha}, \qquad x \neq 0 \tag{3.11}$$

by (2.1), we conclude

$$\limsup_{t \to \infty} t^{-2/\alpha} \ln \tilde{R}_n(t) \leqslant -\varrho\mu^{2/\alpha} \int (1 - e^{-|x|^{-\alpha}})\, dx \tag{3.12}$$

Note that this is also true with $\tilde{N}$ replacing $\tilde{R}_n$, because the different prefactors in the upper bounds (3.6) and (3.8) coincide asymptotically. ▮

Lower Bound. By means of Remark 2.1(iii), the inequality (3.7) reads more explicitly

$$\frac{B}{2\pi} \exp\left(-\varrho \int \left\{1 - \exp\left[-t \int |\phi_n(x-y)|^2\, U(y)\, dy\right]\right\} dx\right) \leqslant \tilde{R}_n(t) \tag{3.13}$$

With the help of the substitution $x \mapsto (\mu t)^{1/\alpha} x$ this yields

$$\begin{aligned} -\varrho\mu^{2/\alpha} \limsup_{t \to \infty} \int \left\{1 - \exp\left[-\int \delta_t(x-y)\, tU((\mu t)^{1/\alpha} y)\, dy\right]\right\} dx \\ \leqslant \liminf_{t \to \infty} t^{-2/\alpha} \ln \tilde{R}_n(t) \end{aligned} \tag{3.14}$$

where we have introduced the one-parameter family $\{\delta_t\}_{t>0} \subset \mathscr{S}(\mathbb{R}^2)$ of probability densities on $\mathbb{R}^2$

$$x \mapsto \delta_t(x) := \frac{B(\mu t)^{2/\alpha}}{2\pi}\, e^{-B(\mu t)^{2/\alpha} x^2/2} [\mathrm{L}_n(B(\mu t)^{2/\alpha} x^2/2)]^2 \tag{3.15}$$

The Fourier representation

$$\delta_t(x) = \frac{1}{(2\pi)^2} \int e^{ikx} e^{-k^2/[2B(\mu t)^{2/\alpha}]} \{\mathrm{L}_n(k^2/[2B(\mu t)^{2/\alpha}])\}^2\, dk \tag{3.16}$$

which may be verified with the help of ref. 20, Eq. (7.377), shows that δ_t approximates Dirac's delta function as $t \to \infty$. Therefore one can check

$$\limsup_{t \to \infty} \int \delta_t(x-y)\, tU((\mu t)^{1/\alpha} y)\, dy \leqslant |x|^{-\alpha}, \qquad x \neq 0 \tag{3.17}$$

using (3.11) and the fact that U is both integrable and square-integrable. According to Fatou's lemma this suffices to arrive at

$$-\varrho\mu^{2/\alpha}\int(1-e^{-|x|^{-\alpha}})\,dx\leqslant\liminf_{t\to\infty}t^{-2/\alpha}\ln\tilde{R}_n(t)\tag{3.18}$$

To obtain the same estimate with $\tilde{N}$ replacing $\tilde{R}_n$ one only has to specialize to $n=0$ and to note that the differing prefactors in (3.5) and (3.7) both become irrelevant on the logarithmic scale. ▌

APPENDIX. PROOFS OF THE BASIC INEQUALITIES

The proofs of the bounds (3.5) and (3.6) for $\tilde{N}$ rely on an approximation formula, which will be presented first.

Approximation A.1. Define for $\Omega\geqslant 0$

$$\hat{V}_{\omega,\Omega}(x):=V_\omega(x)+\frac{\Omega^2}{2}x^2\tag{A.1}$$

Then for t, $\Omega>0$ the Euclidean propagator $\exp[-tH(\hat{V}_{\omega,\Omega})]$ is almost surely of trace class and one has

$$\tilde{N}(t)=\lim_{\Omega\downarrow 0}\frac{\Omega^2 t}{2\pi}\exp(t\varepsilon_0)\,\mathbb{E}[\operatorname{Tr}\{\exp[-tH(\hat{V}_{\omega,\Omega})]\}]\tag{A.2}$$

Proof. According to the Feynman–Kac–Itô formula (see, for example, ref. 44, Theorem 15.5), $\hat{V}_{\omega,\Omega}\in\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)$ and $\hat{V}_{\omega,\Omega}\geqslant 0$ ensure that $\exp[-tH(\hat{V}_{\omega,\Omega})]$ has the integral kernel

$$\exp[-tH(\hat{V}_{\omega,\Omega})](x,y):=\frac{1}{2\pi t}\exp\left(-\frac{(x-y)^2}{2t}\right)\times\int\exp[-S_t(\hat{V}_{\omega,\Omega}\,|\,b)]\,d\mu_{0,x}^{t,y}(b)\tag{A.3}$$

where $\mu_{0,x}^{t,y}$ denotes the probability measure associated with the two-dimensional Brownian bridge from $b(0)=x$ to $b(t)=y$. Here the potential's part of the Euclidean action is given by

$$S_t(\hat{V}_{\omega,\Omega}\,|\,b):=i\int_0^t A(b(s))\,db(s)+\int_0^t\hat{V}_{\omega,\Omega}(b(s))\,ds\tag{A.4}$$

Since the Brownian bridge is a continuous semimartingale (ref. 39, Example V.6.3), the Itô stochastic line integral in (A.4) is well defined (ref. 39, Section II.4).

In ref. 9, Section 6, it is proven that the right-hand side of (A.3) is jointly continuous in (t, x, y), $t>0$, $x, y \in \mathbb{R}^2$, and we will get as a by-product in the proof of (3.6) below that $\exp[-tH(\hat{V}_{\omega,\Omega})]$ is of trace class. Thus (ref. 22, Example X.1.18) we may write

$$\operatorname{Tr}\{\exp[-tH(\hat{V}_{\omega,\Omega})]\} = \frac{1}{2\pi t}\int \exp[-S_t(\hat{V}_{\omega,\Omega}\,|\,b+x)]\, d\mu_{0,0}^{t,0}(b)\, dx \quad \text{(A.5)}$$

where we have performed the rigid shift $b \mapsto b+x$. Due to the translation invariance of V_ω we get with the help of Itô's formula (ref. 39, Corollary to Theorem II.32)

$$\begin{aligned}&\mathbb{E}[\operatorname{Tr}\{\exp[-tH(\hat{V}_{\omega,\Omega})]\}]\\ &\quad = \frac{1}{\Omega^2 t^2}\,\mathbb{E}\left[\int \exp[-S_t(V_\omega\,|\,b) - \Omega^2 t\sigma_t^2(b)/2]\, d\mu_{0,0}^{t,0}(b)\right]\end{aligned} \quad \text{(A.6)}$$

where

$$\sigma_t^2(b) := \int_0^t [b(s)]^2 \frac{ds}{t} - \left[\int_0^t b(s)\frac{ds}{t}\right]^2 \geqslant 0 \quad \text{(A.7)}$$

Since by $V_\omega \geqslant 0$ one has

$$|\exp[-S_t(V_\omega\,|\,b) - \Omega^2 t\sigma_t^2(b)/2]| \leqslant 1 \quad \text{(A.8)}$$

the theorem of dominated convergence is applicable and yields

$$\begin{aligned}&\lim_{\Omega\downarrow 0}\frac{\Omega^2 t}{2\pi}\,\mathbb{E}[\operatorname{Tr}\{\exp[-tH(\hat{V}_{\omega,\Omega})]\}]\\ &\quad = \frac{1}{2\pi t}\,\mathbb{E}\left[\int \exp\{-S_t(V_\omega\,|\,b)\}\, d\mu_{0,0}^{t,0}(b)\right]\end{aligned} \quad \text{(A.9)}$$

Employing again the Feynman–Kac–Itô formula, for $\Omega = 0$, the continuity of the involved integral kernels, and the translation invariance of the random potential, one achieves by (2.19) and (3.2)

$$\frac{1}{2\pi t}\,\mathbb{E}\left[\int e^{-S_t(V_\omega|b)}\, d\mu_{0,0}^{t,0}(b)\right] = e^{-t\varepsilon_0}\tilde{N}(t) \quad \text{(A.10)}$$

which concludes the proof of (A.2). ∎

Proof of (3.5). Let

$$\psi_{p,q}(x) := e^{ip(x+q/2)}\phi_0(x-q) \quad \text{(A.11)}$$

Then $\{\psi_{p,q}\}_{p,q\in\mathbb{R}^2}$ is nothing but the standard overcomplete family of coherent states associated with the Heisenberg–Weyl group generated from the ground state of a two-dimensional isotropic harmonic oscillator. Since $\exp[-tH(\hat{V}_{\omega,\Omega})]$ is almost surely of trace class whenever $t, \Omega > 0$, we may write [see ref. 3, Eq. (1.13) or ref. 46, Theorem A.1.2]

$$\frac{1}{(2\pi)^2}\int \langle \psi_{p,q}, \{\exp[-tH(\hat{V}_{\omega,\Omega})]\}\, \psi_{p,q}\rangle\, dp\, dq$$
$$= \mathrm{Tr}\{\exp[-tH(\hat{V}_{\omega,\Omega})]\} \tag{A.12}$$

Since $H(\hat{V}_{\omega,\Omega})$ is almost surely essentially self-adjoint on $\mathscr{S}(\mathbb{R}^2)$ and $\psi_{p,q}\in\mathscr{S}(\mathbb{R}^2)$, the Jensen–Peierls inequality [ref. 4; ref. 45, Section 8(c)] implies

$$\exp[-t\langle \psi_{p,q}, H(\hat{V}_{\omega,\Omega})\,\psi_{p,q}\rangle]$$
$$\leqslant \langle \psi_{p,q}, \{\exp[-tH(\hat{V}_{\omega,\Omega})]\}\, \psi_{p,q}\rangle \tag{A.13}$$

Due to the translation invariance of the random potential, we get by inserting (A.13) into (A.12) after some calculation

$$\frac{2\pi}{\Omega^2 t}\exp\left(-\frac{\Omega^2 t}{B}\right)\frac{\exp(-t\varepsilon_0)}{2\pi t}\,\mathbb{E}[\exp(-t\langle\phi_0, V_\omega\phi_0\rangle)]$$
$$\leqslant \mathbb{E}[\mathrm{Tr}\{\exp[-tH(\hat{V}_{\omega,\Omega})]\}] \tag{A.14}$$

With the help of (A.2) this proves (3.5). ∎

Proof of (3.6). First note that for $\Omega > 0$ the operators $H(0)$, $\hat{V}_{\omega,\Omega}$, and $H(\hat{V}_{\omega,\Omega})$ are almost surely essentially self-adjoint on $\mathscr{S}(\mathbb{R}^2)$ and non-negative. Furthermore, $\exp[-tH(0)/2]\exp(-t\hat{V}_{\omega,\Omega}/2)$ is almost surely Hilbert–Schmidt, since (ref. 40, Theorem VI.23)

$$\int \left|\exp\left[\frac{-tH(0)}{2}\right](x, y)\exp\left[\frac{-t\hat{V}_{\omega,\Omega}(y)}{2}\right]\right|^2 dx\, dy$$
$$= \frac{B}{4\pi\sinh(t\varepsilon_0)}\int \exp[-t\hat{V}_{\omega,\Omega}(y)]\, dy < \infty \tag{A.15}$$

Therefore, we can use the Golden–Thompson inequality in the version given in the Corollary to Theorem XIII.103 in ref. 42 to conclude that $\exp[-tH(\hat{V}_{\omega,\Omega})]$ is almost surely of trace class and its trace is bounded from above by the squared Hilbert–Schmidt norm (A.15). Taking the expectation with respect to the random potential, we find that this shows

$$\mathbb{E}[\operatorname{Tr}\{\exp[-tH(\hat{V}_{\omega,\Omega})]\}] \leqslant \frac{2\pi}{\Omega^2 t} \frac{B}{4\pi \sinh(t\varepsilon_0)} \mathbb{E}[\exp\{-tV_\omega(0)\}] \quad (A.16)$$

Using (A.2) once again, we have the proof of (3.6). ∎

As a preliminary to the proofs of (3.7) and (3.8) we show that the nth restricted integrated density of states R_n is independent of $x \in \mathbb{R}^2$. This is achieved with the help of the unitary magnetic-translation operator W_x defined by

$$(W_x\psi)(y) := e^{(iB/2)(y_1x_2 - y_2x_1)}\psi(y-x), \qquad \psi \in L^2(\mathbb{R}^2) \qquad (A.17)$$

because on the one hand the eigenprojectors E_n are left invariant under its action, that is,

$$W_x^\dagger E_n W_x = E_n \qquad (A.18)$$

and on the other hand the potential is simply shifted,

$$W_x^\dagger V_\omega W_x = V_\omega \circ T_x, \qquad T_x y := y + x \qquad (A.19)$$

Using these properties together with the translation invariance of the random potential, we see that the right-hand side of (2.20) is independent of $x \in \mathbb{R}^2$. For a complete discussion of the magnetic-translation group associated with $H(0)$ see refs. 52 and 25.

Proof of (3.7). The combination of the definition (2.20) for $x=0$ with (3.3) and (3.4) gives a more explicit expression

$$\tilde{R}_n(t) = \frac{B}{2\pi} \mathbb{E}[\langle \phi_n, e^{-tE_n V_\omega E_n}\phi_n\rangle] \qquad (A.20)$$

for the shifted Laplace transform of the nth restricted integrated density of states. Since $\phi_n \in \mathscr{S}(\mathbb{R}^2)$ and $E_n V_\omega E_n$ is almost surely essentially self-adjoint on $\mathscr{S}(\mathbb{R}^2)$, we may use the Jensen–Peierls inequality [ref. 4; ref. 45, Section 8(c)] and the fact that $E_n \phi_n = \phi_n$ to estimate

$$e^{-t\langle\phi_n, V_\omega \phi_n\rangle} \leqslant \langle \phi_n, e^{-tE_n V_\omega E_n}\phi_n\rangle \qquad (A.21)$$

The insertion of (A.21) into (A.20) completes the proof of (3.7). ∎

For the proof of (3.8) we will use again an approximation formula which we single out as follows.

Approximation A.2. Define the centered Poisson potential, truncated outside a disk of radius $r > 0$ about the origin,

$$\check{V}_{\omega,r}(x) := \Theta(r - |x|)\{V_\omega(x) - \mathbb{E}[V_\omega(0)]\} \tag{A.22}$$

Then

$$\lim_{r\to\infty} \frac{1}{\pi r^2} \int_{|x|\leqslant r} \mathbb{E}[\{E_n[\exp(-tE_n \check{V}_{\omega,r} E_n)]\, E_n\}(x,x)]\, dx$$
$$= (\exp\{t\mathbb{E}[V_\omega(0)]\})\, \tilde{R}_n(t) \tag{A.23}$$

Proof. First we use for the magnetic-translation operator (A.17) the properties (A.18) and (A.19) with $\check{V}_{\omega,r}$ replacing V_ω, the definition (3.4) of ϕ_n, and the translation invariance of V_ω to rewrite

$$\frac{1}{\pi r^2} \int_{|x|\leqslant r} \mathbb{E}[\{E_n[\exp(-tE_n \check{V}_{\omega,r} E_n)]\, E_n\}(x,x)]\, dx$$
$$= \frac{B}{2\pi^2} \int_{|\zeta|\leqslant 1} \mathbb{E}[\langle \phi_n, [\exp(-tE_n \bar{V}_{\omega,r,\zeta} E_n)]\, \phi_n\rangle]\, d\zeta \tag{A.24}$$

Here we have introduced

$$\bar{V}_{\omega,r,\zeta}(x) := \Theta\left(1 - \left|\zeta - \frac{x}{r}\right|\right)\{V_\omega(x) - \mathbb{E}[V_\omega(0)]\} \tag{A.25}$$

As a next step we claim that for all $|\zeta| < 1$ and almost all ω

$$E_n \bar{V}_{\omega,r,\zeta} E_n \xrightarrow{r\to\infty} E_n(V_\omega - \mathbb{E}[V_\omega(0)])\, E_n \tag{A.26}$$

in the strong resolvent sense. Since $E_n \bar{V}_{\omega,r,\zeta} E_n$ is almost surely essentially self-adjoint on $\mathscr{S}(\mathbb{R}^2)$ for all $r > 0$ and so is the right-hand side of (A.26), it is sufficient (ref. 40, Theorem VIII.25) to check that for all $\psi \in \mathscr{S}(\mathbb{R}^2)$ and almost all ω

$$\|E_n(\bar{V}_{\omega,r,\zeta} - V_\omega + \mathbb{E}[V_\omega(0)])\, E_n\psi\|_2 \xrightarrow{r\to\infty} 0 \tag{A.27}$$

Since E_n is bounded and maps $\mathscr{S}(\mathbb{R}^2)$ into itself, this follows from

$$\int |\{\bar{V}_{\omega,r,\zeta}(x) - V_\omega(x) + \mathbb{E}[V_\omega(0)]\}\, \psi(x)|^2\, dx$$
$$\leqslant \int_{|x| > r(1-|\zeta|)} |\{V_\omega(x) - \mathbb{E}[V_\omega(0)]\}\, \psi(x)|^2\, dx \xrightarrow{r\to\infty} 0 \tag{A.28}$$

for all $\psi \in \mathscr{S}(\mathbb{R}^2)$, $|\zeta| < 1$. The last inequality is due to the estimate $\Theta(|\zeta - x/r| - 1) \leqslant \Theta(|x| - r(1 - |\zeta|))$ and its right-hand side vanishes as $r \to \infty$, because the mapping (2.8) is almost surely polynomially bounded.

The strong resolvent convergence (A.26) now implies (ref. 40, Theorem VIII.20), together with (A.20),

$$\lim_{r \to \infty} \mathbb{E}[\langle \phi_n, e^{-tE_n \bar{V}_{\omega, r, \zeta} E_n} \phi_n \rangle] = \frac{2\pi}{B} e^{t\mathbb{E}[V_\omega(0)]} \tilde{R}_n(t) \tag{A.29}$$

From $\bar{V}_{\omega, r, \zeta}(x) \geqslant -\mathbb{E}[V_\omega(0)]$ one has

$$\langle \phi_n, e^{-tE_n \bar{V}_{\omega, r, \zeta} E_n} \phi_n \rangle \leqslant e^{t\mathbb{E}[V_\omega(0)]} \tag{A.30}$$

almost surely. Therefore one can use the dominated-convergence theorem, (A.29), and (A.24) to obtain (A.23). ∎

Proof of (3.8). Let $\check{V}_{\omega, r}$ be as given in (A.22) and $0 < r < \infty$. We first note that $E_n \check{V}_{\omega, r} E_n$ is almost surely of trace class. This can be seen [ref. 40, Theorem VI.22(h)], for example, by noting that both $E_n |\check{V}_{\omega, r}|^{1/2}$ and $\operatorname{sgn}(\check{V}_{\omega, r}) |\check{V}_{\omega, r}|^{1/2} E_n$ are almost surely Hilbert–Schmidt because (ref. 40, Theorem VI.23)

$$\int |E_n(x, y) |\check{V}_{\omega, r}(y)|^{1/2}|^2 \, dx \, dy = \frac{B}{2\pi} \int |\check{V}_{\omega, r}(y)| \, dy < \infty \tag{A.31}$$

Therefore

$$\exp(-tE_n \check{V}_{\omega, r} E_n) - 1 = E_n \check{V}_{\omega, r} E_n \sum_{k=1}^{\infty} \frac{(-t)^k}{k!} (E_n \check{V}_{\omega, r} E_n)^{k-1} \tag{A.32}$$

is of trace class, because it is the product of a trace-class operator and a norm-convergent sum [ref. 40, Theorem VI.19(c)]. Thus we have

$$\begin{aligned} &\int \mathbb{E}[\{E_n[\exp(-tE_n \check{V}_{\omega, r} E_n) - 1] \, E_n\}(x, x)] \, dx \\ &\quad = \mathbb{E}[\operatorname{Tr}\{E_n[\exp(-tE_n \check{V}_{\omega, r} E_n) - 1] \, E_n\}] \end{aligned} \tag{A.33}$$

due to the continuity of the integral kernel (ref. 22, Example X.1.18); see Remark 2.2(iii).

Analogous to the arguments in the proof of (3.7), we may rewrite the integrand in the approximation formula (A.23) as a scalar product, use the

Jensen–Peierls inequality, and in a next step the Jensen inequality together with $\mathbb{E}[\check{V}_{\omega,r}(x)]=0$ to show

$$\mathbb{E}[\{E_n[\exp(-tE_n\check{V}_{\omega,r}E_n)-1]\,E_n\}(x,x)] \geqslant \frac{B}{2\pi}\,\mathbb{E}\left[\exp\left\{-\frac{2\pi}{B}\,t(E_n\check{V}_{\omega,r}E_n)(x,x)\right\}-1\right]\geqslant 0 \qquad \text{(A.34)}$$

Thanks to (A.34), (A.33) and the approximation formula (A.23) can be combined to yield the preliminary estimate

$$(\exp\{t\mathbb{E}[V_\omega(0)]\})\,\tilde{R}_n(t)-\frac{B}{2\pi} \leqslant \limsup_{r\to\infty}\frac{1}{\pi r^2}\,\mathbb{E}[\mathrm{Tr}\{E_n[\exp(-tE_n\check{V}_{\omega,r}E_n)-1]\,E_n\}] \qquad \text{(A.35)}$$

Finally, we use the Jensen-Peierls inequality in the version of Berezin[4] [see also ref. 45, Section 8(c)], which implies

$$\mathrm{Tr}\{E_n[\exp(-tE_n\check{V}_{\omega,r}E_n)-1]\,E_n\}\leqslant \mathrm{Tr}\{E_n[\exp(-t\check{V}_{\omega,r})-1]\,E_n\} \qquad \text{(A.36)}$$

This is justified, since both $E_n\check{V}_{\omega,r}E_n$ and $\check{V}_{\omega,r}$ are almost surely essentially self-adjoint on $\mathcal{S}(\mathbb{R}^2)$, $E_n\mathcal{S}(\mathbb{R}^2)$ is dense in $E_n\mathrm{L}^2(\mathbb{R}^2)$, and the right-hand side is well defined. The latter can be seen by an argument analogous to that at the beginning of this proof employing $[\exp(-t\check{V}_{\omega,r})-1]\in\mathrm{L}^1(\mathbb{R}^2)$. Due to the continuity of the integral kernel of $E_n[\exp(-t\check{V}_{\omega,r})-1]\,E_n$ we are allowed (ref. 22, Example X.1.18) to calculate

$$\begin{aligned}\mathbb{E}[\mathrm{Tr}\{E_n[\exp(-t\check{V}_{\omega,r})-1]\,E_n\}] &=\frac{B}{2\pi}\,\mathbb{E}\left[\int\{\exp[-t\check{V}_{\omega,r}(x)]-1\}\,dx\right]\\ &=\frac{Br^2}{2}\,(\exp\{t\mathbb{E}[V_\omega(0)]\}\,\mathbb{E}[\exp\{-tV_\omega(0)\}]-1)\end{aligned} \qquad \text{(A.37)}$$

To finish the proof of (3.8), we only have to insert (A.36) into (A.37) and perform the limit with the help of (A.37). ▮

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REFERENCES

1. T. Ando, A. B. Fowler, and F. Stern, Electronic properties of two-dimensional systems, *Rev. Mod. Phys.* **54**:437 (1982).
2. J. Bellissard, A. van Elst, and H. Schulz-Baldes, The noncommutative geometry of the quantum Hall effect, *J. Math. Phys.* **35**:5373 (1994).
3. F. A. Berezin, *Mat. Sbornik* **86**:578 (1971) [English transl., Wick and anti-Wick operator symbols, *Math. USSR Sbornik* **15**:577 (1971)].
4. F. A. Berezin, *Mat. Sbornik* **88**:268 (1972) [English transl., Convex operator functions, *Math. USSR Sbornik* **17**:269 (1972)].
5. E. Brézin, D. J. Gross, and C. Itzykson, Density of states in the presence of a strong magnetic field and random impurities, *Nucl. Phys. B* **235**:24 (1984).
6. K. Broderix, N. Heldt, and H. Leschke, Partition function and the density of states for an electron in the plane subjected to a random potential and a magnetic field, *Phys. Rev. B* **40**:7479 (1989).
7. K. Broderix, N. Heldt, and H. Leschke, Exact results on Landau-level broadening, *J. Phys. A* **24**:L825 (1991).
8. K. Broderix, D. Hundertmark, and H. Leschke, Self-averaging, decomposition and asymptotic properties of the density of states for random Schrödinger operators with constant magnetic field, in *Path Integrals from meV to MeV, Tutzing '92*, H. Grabert, A. Inomata, L. S. Schulman, and U. Weiss, eds. (World Scientific, Singapore, 1993), p. 98.
9. K. Broderix, D. Hundertmark, and H. Leschke, Continuity properties of Schrödinger semigroups with magnetic fields, Preprint, Universität Göttingen, Ruhr-Universität Bochum, and Universität Erlangen-Nürnberg (1995).
10. K. Broderix, D. Hundertmark, W. Kirsch, and H. Leschke, On the density of states for random Schrödinger operators with constant magnetic fields, In preparation.
11. R. Carmona and J. Lacroix, *Spectral Theory of Random Schrödinger Operators* (Birkhäuser, Boston, 1990).
12. J. M. Combes and P. D. Hislop, Landau Hamiltonians with random potentials: Localization and the density of states, Preprint, Université de Toulon et du Var, and University of Kentucky-Lexington (1994).
13. H. L. Cycon, R. G. Froese, W. Kirsch, and B. Simon, *Schrödinger Operators with Application to Quantum Mechanics and Global Geometry* (Springer, Berlin, 1987).
14. J.-D. Deuschel and D. W. Stroock, *Large Deviations* (Academic Press, Boston, 1989).
15. M. D. Donsker and S. R. S. Varadhan, Asymptotics for the Wiener sausage, *Commun. Pure. Appl. Math.* **28**:525 (1975); Erratum, *Commun. Pure Appl. Math.* **28**:677 (1975).
16. T. C. Dorlas, N. Macris, and J. V. Pulé, Localisation in a single-band approximation to random Schroedinger operators in a magnetic field, Preprint, University College Swansea, Ecole Polytechnique Fédérale de Lausanne, and University College Dublin (1995).
17. T. C. Dorlas, N. Macris, and J. V. Pulé, Localisation in single Landau bands. Preprint, University College Swansea, Ecole Polytechnique Fédérale de Lausanne, and University College Dublin (1995).

18. R. Friedberg and J. M. Luttinger, Density of electronic energy levels in disordered systems, *Phys. Rev. B* **12**:4460 (1975).
19. A. I. Gusev, *Mat. Sbornik* **104**:207 (1977) [English transl., The state density and other spectral invariants of selfadjoint elliptic operators with random coefficients, *Math. USSR Sbornik* **33**:185 (1977)].
20. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, corrected and enlarged ed. (Academic Press, Orlando, Florida, 1980).
21. M. Janssen, O. Viehweger, U. Fastenrath, and J. Hajdu, *Introduction to the Theory of the Integer Quantum Hall Effect* (VCH, Weinheim, 1994).
22. T. Kato, *Perturbation Theory for Linear Operators*, 2nd ed., 2nd corrected printing (Springer, Berlin, 1984).
23. W. Kirsch, Random Schrödinger operators. A course, in *Schrödinger Operators*, H. Holden and A. Jensen, eds. (Springer, Berlin, 1989), p. 264.
24. W. Kirsch and F. Martinelli, On the ergodic properties of the spectrum of general random operators, *J. Reine Angew. Math.* **334**:141 (1982).
25. A. Klein and J. F. Perez, On the density of states for random potentials in the presence of a uniform magnetic field, *Nucl. Phys. B* **251**:199 (1985).
26. I. V. Kukushkin, S. V. Meshkov, and V. B. Timofeev, *Usp. Fiz. Nauk* **155**:219 (1988) [English transl., Two-dimensional electron density of states in a transverse magnetic field, *Sov. Phys. Usp.* **31**:511 (1988)].
27. L. Landau, Diamagnetismus der Metalle, *Z. Phys.* **64**:629 (1930).
28. H. Leinfelder, Gauge invariance of Schrödinger operators and related spectral properties, *J. Operator Theory* **9**:163 (1983).
29. I. M. Lifshitz, *Usp. Fiz. Nauk* **83**:617 (1964) [English transl., Energy spectrum structure and quantum states of disordered condensed systems, *Sov. Phys. Usp.* **7**:549 (1965); The energy spectrum of disordered systems, *Adv. Phys.* **13**:483 (1964)].
30. I. M. Lifshitz, *Zh. Eksp. Teor. Fiz.* **53**:743 (1967) [English transl., Theory of fluctuation levels in disordered systems, *Sov. Phys. JETP* **26**:462 (1968)].
31. I. M. Lifshits, S. A. Gredeskul, and L. A. Pastur, *Introduction to the Theory of Disordered Systems* (Wiley, New York, 1988).
32. J. M. Luttinger, New variational method with applications to disordered systems, *Phys. Rev. Lett.* **37**:609 (1976).
33. J. M. Luttinger and R. Waxler, Low energy density of states in several disordered systems, *Ann. Phys.* (N.Y.) **175**:319 (1987).
34. N. Macris and J. V. Pulé, Density of states of random Schrödinger operators with a uniform magnetic field, *Lett. Math. Phys.* **24**:307 (1992).
35. H. Matsumoto, On the integrated density of states for the Schrödinger operators with certain random electromagnetic potentials, *J. Math. Soc. Japan* **45**:197 (1993).
36. S. Nakao, On the spectral distribution of the Schrödinger operator with random potential, *Jpn. J. Math.* **3**:111 (1977).
37. L. A. Pastur, *Teor. Mat. Fiz.* **32**:88 (1977) [English transl., Behavior of some Wiener integrals as $t \to \infty$ and the density of states of Schrödinger equations with random potential, *Theor. Math. Phys.* **32**:615 (1977)].
38. L. Pastur and A. Figotin, *Spectra of Random and Almost-Periodic Operators* (Springer, Berlin, 1992).
39. P. Protter, *Stochastic Integration and Differential Equations: A New Approach*, 2nd corrected printing (Springer, Berlin, 1992).
40. M. Reed and B. Simon, *Methods of Modern Mathematical Physics I: Functional Analysis*, rev. and enlarged ed. (Academic Press, New York, 1980).
41. M. Reed and B. Simon, *Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness* (Academic Press, New York, 1975).

42. M. Reed and B. Simon, *Methods of Modern Mathematical Physics IV: Analysis of Operators* (Academic Press, New York, 1978).
43. B. I. Shklovskii and A. L. Efros, *Electronic Properties of Doped Semiconductors* (Springer, Berlin, 1984).
44. B. Simon, *Functional Integration and Quantum Physics* (Academic Press, New York, 1979).
45. B. Simon, *Trace Ideals and Their Applications* (Cambridge University Press, Cambridge, 1979).
46. B. Simon, The classical limit of quantum partition functions, *Commun. Math. Phys.* **71**:247 (1980).
47. B. Simon, Schrödinger semigroups, *Bull. Am. Math. Soc.* **7**:447 (1982); Erratum, *Bull. Am. Math. Soc.* **11**:426 (1984).
48. M. Stone, ed., *Quantum Hall Effect* (World Scientific, Singapore, 1992).
49. N. Ueki, On spectra of random Schrödinger operators with magnetic fields, *Osaka J. Math.* **31**:177 (1994).
50. Wei-Min Wang, Asymptotic expansion for the density of states of the random Schrödinger operator with a magnetic field, Preprint, Université de Paris-Sud (1994).
51. F. Wegner, Exact density of states for lowest Landau level in white noise potential. Superfield representation for interacting systems, *Z. Phys. B* **51**:279 (1983).
52. J. Zak, Magnetic translation group, *Phys. Rev.* **134**:A1602 (1964).